1 $\displaystyle\int_0^{2\pi} \cos(x)\,dv$

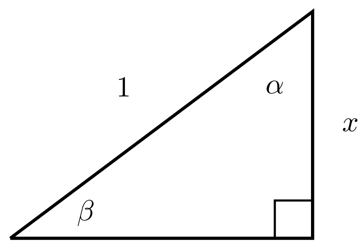
Be careful! The variable to be integrated is v, not x.

$$= \cos(x)\int_0^{2\pi} dv = \boxed{2\pi\cos(x)}$$

---

2 $\displaystyle\int_{-1/2}^{3/4} \arccos(x) + \arcsin(x)\,dx$

Observe from the figure below that $\arccos(x) + \arcsin(x) = \alpha + \beta = \dfrac{\pi}{2}$



$$\int_{-1/2}^{3/4} \arccos(x) + \arcsin(x)\,dx = \frac{\pi}{2}\int_{-1/2}^{3/4} dx = \boxed{\frac{5\pi}{8}}$$

---

3 $\displaystyle\int_0^{\infty} x^4 e^{-x}\,dx$

Although we can manually integrate by parts, generalizing yields an interesting result.

Let $I_n = \displaystyle\int_0^{\infty} x^n e^{-x}\,dx$. Integrate by parts to express I_n in terms of itself.

$$I_n = \int_0^{\infty} x^n e^{-x}\,dx = -x^n e^{-x}\Big|_0^{\infty} + n\int_0^{\infty} x^{n-1}e^{-x}\,dx = nI_{n-1}$$

We see that $I_n = nI_{n-1}$. There is one other operation that does this– the factorial! This is actually the generalization of the factorial to complex numbers.

$$I_4 = 4! = \boxed{24}$$

---

4 $\displaystyle\int_0^{1} \frac{x^6}{x^2+1}\,dx$

Recall that $n^3 + 1 = (n+1)(n^2 - n + 1)$

$$= \int_0^1 \frac{(x^2)^3 + 1 - 1}{x^2+1}\,dx = \int_0^1 \frac{(x^2)^3+1}{x^2+1} - \frac{1}{x^2+1}\,dx = \int_0^1 (x^2)^2 - x^2 + 1 - \frac{1}{x^2+1}\,dx$$

Integrate and evaluate.

$$= \frac{1}{5}x^5 - \frac{1}{3}x^3 + x - \arctan(x)\Big|_0^1 = \boxed{\frac{13}{15} - \frac{\pi}{4}}$$

$\boxed{\textbf{5}}$ $\displaystyle\int_0^{\sinh(1)} \operatorname{arsinh}(x)\,dx$

Inverse substitution: let $\sinh(u) = x$ and $\cosh(u)\,du = dx$, then integrate by parts.

$$= \int_0^1 u\cosh(u)\,du = u\sinh(u) - \cosh(u)\Big|_0^1 = \sinh(1) - \cosh(1) + 1 = \boxed{1 - \frac{1}{e}}$$

$\boxed{\textbf{6}}$ $\displaystyle\int_0^{\sqrt[4]{15}} \sqrt[4]{x^{16} + x^{12}}\,dx$

Factor x^{12} out of the radical, then let $u = x^4 + 1$ and $du = 4x^3\,dx$.

$$= \int_0^{\sqrt[4]{15}} \sqrt[4]{x^{12}(x^4+1)}\,dx = \int_0^{\sqrt[4]{15}} x^3\sqrt[4]{x^4+1}\,dx = \frac{1}{4}\int_1^{16} \sqrt[4]{u}\,du = \frac{1}{4}\cdot\frac{4}{5}u^{5/4}\Big|_1^{16} = \boxed{\frac{31}{5}}$$

$\boxed{\textbf{7}}$ $\displaystyle\int_0^2 x(1-x)^4\,dx$

Let $u = 1 - x$ and $du = -dx$. This is technically an application of Queen's Rule.

$$= -\int_1^{-1} (1-u)u^4\,du = \int_{-1}^1 u^4 - u^5\,du$$

Because the new bounds are opposites, we can double the u^4 term because it is even, and ignore the u^5 term because it is odd.

$$= 2\int_0^1 u^4\,du = \frac{2}{5}u^5\Big|_0^1 = \boxed{\frac{2}{5}}$$

$\boxed{\textbf{8}}$ $\displaystyle\int_1^2 x\operatorname{arcsec}(x)\,dx$

Method 1: I memorized the derivative of arcsecant! $\left(\dfrac{d}{dx}\operatorname{arcsec}(x) = \dfrac{1}{x\sqrt{x^2-1}}\right)$

Directly integrate by parts:

$$= \frac{1}{2}x^2\operatorname{arcsec}(x)\Big|_1^2 - \frac{1}{2}\int_1^2 \frac{x}{\sqrt{x^2-1}}\,dx = \frac{2\pi}{3} - \frac{1}{2}\sqrt{x^2-1}\Big|_1^2 = \boxed{\frac{2\pi}{3} - \frac{\sqrt{3}}{2}}$$

Method 2: I didn't memorize the derivative of arcsecant...

Inverse substitution: let $x = \sec(u)$ and $dx = \sec(u)\tan(u)\,du$, then integrate by parts.

$$= \int_0^{\pi/3} x\sec^2(x)\tan(u)\,du = \frac{1}{2}x\sec^2(u)\Big|_0^{\pi/3} - \frac{1}{2}\int_0^{\pi/3}\sec^2(u)\,du = \frac{2\pi}{3} - \frac{1}{2}\tan(u)\Big|_0^{\pi/3} = \boxed{\frac{2\pi}{3} - \frac{\sqrt{3}}{2}}$$

---

9 $\displaystyle\int_0^\pi \cos^6(x)\,dx$

This integral can be evaluated using the Wallis product; for those unfamiliar, a short explanation is given below.

Let $I_n = \displaystyle\int_0^\pi \cos^n(x)\,dx$. Construct a recurrence relation using integration by parts.

$$I_n = \int_0^\pi \cos(x)\cos^{n-1}(x)\,dx = \sin(x)\cos^{n-1}(x)\Big|_0^\pi + (n-1)\int_0^\pi \sin^2(x)\cos^{n-2}(x)\,dx$$

The component outside the integral evaluates to 0, so I_n can be expressed in terms of itself:

$$I_n = (n-1)\int_0^\pi (1-\cos^2(x))\cos^{n-2}(x)\,dx = (n-1)\int_0^\pi \cos^{n-2}(x) - \cos^n(x)\,dx = (n-1)(I_{n-2} - I_n)$$

Solving for I_n gives the recurrence relation $I_n = \dfrac{n-1}{n} I_{n-2}$.

The base case I_0 is simply $I_0 = \displaystyle\int_0^\pi dx = \pi$, so we can evaluate the given integral I_6:

$$\int_0^\pi \cos^6(x)\,dx = I_6 = \frac{6-1}{6} I_4 = \frac{5}{6}\cdot\frac{4-1}{4} I_2 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{2-1}{2} I_0 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\pi = \boxed{\frac{5\pi}{16}}$$

---

10 $\displaystyle\int_1^\infty \frac{x+1}{x^3+x}\,dx$

Factor out x from the denominator and split the fraction.

$$= \int_1^\infty \frac{x+1}{x(x^2+1)}\,dx = \int_1^\infty \frac{x}{x(x^2+1)} + \frac{1}{x(x^2+1)}\,dx = \int_1^\infty \frac{1}{x^2+1} + \frac{1}{x(x^2+1)}\,dx$$

The left component is straightforward to evaluate.

$$\int_1^\infty \frac{1}{x^2+1}\,dx = \arctan(x)\Big|_1^\infty = \frac{\pi}{4}$$

For the right component, factor out x^2 from the denominator, then let $u = 1 + \dfrac{1}{x^2}$ and $du = -\dfrac{2}{x^3}\,dx$

$$\int_1^\infty \frac{1}{x(x^2+1)}\,dx = \int_1^\infty \frac{1}{x^3\left(1+\frac{1}{x^2}\right)}\,dx = -\frac{1}{2}\int_2^1 \frac{1}{u}\,du = \frac{1}{2}\ln(u)\Big|_1^2 = \frac{1}{2}\ln(2)$$

Remember to combine the components in the end!

$$\int_1^\infty \frac{x+1}{x^3+x}\,dx = \boxed{\frac{\pi}{4} + \frac{1}{2}\ln(2)}$$

---

11 $\displaystyle\int_1^4 e^{2x}\left(2\ln(x)+\frac{1}{x}\right)\,dx$

Reverse product rule: note that $\dfrac{d}{dx}\left[e^{2x}f(x)\right] = 2e^{2x}f(x)+e^{2x}f'(x) = e^{2x}\left[2f(x)+f'(x)\right]$, which means that $f(x)=\ln(x)$ in this case.

$$= e^{2x}\ln(x)\Big|_1^4 = \boxed{2e^8\ln(2)}$$

---

12 $\displaystyle\int_0^{\pi/4}\frac{\sin(x)+8\cos(x)}{2\sin(x)+\cos(x)}\,dx$

We want to write the integrand as a linear combination of two components (one of which cancels out).

$$\frac{\sin(x)+8\cos(x)}{2\sin(x)+\cos(x)} = a\cdot\frac{2\cos(x)-\sin(x)}{2\sin(x)+\cos(x)} + b\cdot\frac{2\sin(x)+\cos(x)}{2\sin(x)+\cos(x)}$$

We can go ahead and integrate these components.

$$\int_0^{\pi/4} a\cdot\frac{2\cos(x)-\sin(x)}{2\sin(x)+\cos(x)}+b\,dx = a\ln(2\sin(x)+\cos(x))+bx\Big|_0^{\pi/4} = a\ln\left(\frac{3}{\sqrt{2}}\right)+b\cdot\frac{\pi}{4}$$

To find the constants a and b, we will need to build a system of equations, with is totally something you should expect to do in an integration bee.
Since the denominators are the same, focus on the numerators and distribute the constants.

$$\sin(x)+8\cos(x) = a[2\cos(x)-\sin(x)]+b[2\sin(x)+\cos(x)]$$
$$\sin(x)+8\cos(x) = 2a\cos(x)-a\sin(x)+2b\sin(x)+b\cos(x)$$

Match the sines and cosines and solve the resulting system of equations.

$$\begin{cases}2a\cos(x)+b\cos(x) & =8\cos(x)\\ -a\sin(x)+2b\sin(x) & =\sin(x)\end{cases} \implies \begin{cases}2a+b & =8\\ -a+2b & =1\end{cases} \implies \begin{cases}a=3\\ b=2\end{cases}$$

We can now substitute the values of a and b in our answer:

$$a\ln\left(\frac{3}{\sqrt{2}}\right)+b\cdot\frac{\pi}{4} = \boxed{3\ln\left(\frac{3}{\sqrt{2}}\right)+\frac{\pi}{2}}$$

---

13 $\displaystyle\int_0^{2\pi}\sin(\sin(x)-x)\,dx$

King's Rule: let $u=2\pi-x$ and $du=-dx$. Note that $\sin(2\pi-x)=-\sin(x)$.

$$I = -\int_{2\pi}^{0}\sin(\sin(2\pi-u))-(2\pi-u))\,du = \int_0^{2\pi}\sin(-\sin(u)-2\pi+u)\,du$$

By simplifying, we find that $I=-I$, which means that $I=0$.

$$= \int_0^{2\pi}\sin(2\pi-(\sin(u)-u))\,du = -\int_0^{2\pi}\sin(\sin(u)-u)\,du = -I = \boxed{0}$$

---

14 $\displaystyle\int_0^{\pi/2} \frac{\sin^2(x)}{1+\sin(2x)}\,dx$

Use trig identities to expand and factor the denominator.

$$I = \int_0^{\pi/2} \frac{\sin^2(x)}{\sin^2(x)+\cos^2(x)+2\sin(x)\cos(x)}\,dx = \int_0^{\pi/2} \frac{\sin^2(x)}{(\sin(x)+\cos(x))^2}\,dx$$

King's Rule: let $u = \frac{\pi}{2} - x$ and $du = -dx$. This has the effect of converting sines to cosines and vice versa.

$$I = -\int_{\pi/2}^{0} \frac{\sin^2(\frac{\pi}{2}-x)}{(\sin(\frac{\pi}{2}-x)+\cos(\frac{\pi}{2}-x))^2}\,dx = \int_0^{\pi/2} \frac{\cos^2(x)}{(\cos(x)+\sin(x))^2}\,dx$$

Since these integrals are equivalent and have the same bounds, we can add them together and halve the result:

$$I = \frac{1}{2}\int_0^{\pi/2} \frac{\sin^2(x)}{(\sin(x)+\cos(x))^2} + \frac{\cos^2(x)}{(\cos(x)+\sin(x))^2}\,dx = \frac{1}{2}\int_0^{\pi/2} \frac{1}{(\sin(x)+\cos(x))^2}\,dx$$

Multiply both sides by $\sec^2(x)$, then let $u = \tan(x)+1$ and $du = \sec^2(x)\,dx$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sec^2(x)}{(\tan(x)+1)^2}\,dx = \frac{1}{2}\int_1^{\infty} \frac{1}{u^2}\,du = -\frac{1}{2u}\bigg|_1^{\infty} = \boxed{\frac{1}{2}}$$

---

15 $\displaystyle\int_0^{45} \frac{2x-x^2}{x^2+2025e^x}\,dx$

Add $2025e^x - 2025e^x$ to the numerator, then decompose the fraction.

$$= \int_0^{45} \frac{2x+2025e^x-x^2-2025e^x}{x^2+2025e^x}\,dx = \int_0^{45} \frac{2x+2025e^x}{x^2+2025e^x} - 1\,dx = \ln(x^2+2025e^x) - x\bigg|_0^{45}$$

There are two equivalent ways to express the answer:

$$= \boxed{\ln(1+e^{45}) - 45} = \boxed{\ln(e^{-45}+1)}$$

---

16 $\displaystyle\int_0^{1} \frac{1}{\sqrt{-\ln(x)}}\,dx$

Let $u = \sqrt{-\ln(x)}$ and $du = -\dfrac{1}{2x\sqrt{-\ln(x)}}$ to transform the integral into the Gaussian integral.

$$\int_0^1 \frac{2x}{2x\sqrt{-\ln(x)}}\,dx = -2\int_{\infty}^{0} e^{-u^2}\,du = 2\int_0^{\infty} e^{-u^2}\,du = \boxed{\sqrt{\pi}}$$

---

17 $\int_0^\infty \sin(x^2)\,dx$

This integral is known as the *Fresnel integral*. If you haven't memorized it, here is a quick but non-rigorous derivation involving the Gaussian integral:

$$\int_0^\infty e^{ix^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{-i}} = \frac{\sqrt{\pi i}}{2} = \frac{\sqrt{\pi}}{2}e^{\pi i/4}$$

Expanding the complex exponential using Euler's identity gives:

$$\int_0^\infty \cos(x^2) + i\sin(x^2)\,dx = \frac{\sqrt{\pi}}{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = \frac{\sqrt{\pi}}{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \frac{1}{2}\sqrt{\frac{\pi}{2}}(1+i)$$

Matching the real and imaginary components gives:

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} = \boxed{\sqrt{\frac{\pi}{8}}}$$

---

18 $\int_0^{\pi/2} \frac{9\sin(x)+5\cos(x)}{\sin(x)+\cos(x)}\,dx$

King's rule- let $u = \frac{\pi}{2} - x$ and $du = -dx$ to switch the sines and cosines.

$$I = -\int_{\pi/2}^{0} \frac{9\sin(\frac{\pi}{2}-u)+5\cos(\frac{\pi}{2}-u)}{\sin(\frac{\pi}{2}-u)+\cos(\frac{\pi}{2}-u)}\,du = \int_0^{\pi/2} \frac{9\cos(u)+5\sin(u)}{\cos(u)+\sin(u)}\,du$$

Add to the original integral and halve the result.

$$I = \frac{1}{2}\int_0^{\pi/2} \frac{9\sin(x)+5\cos(x)}{\sin(x)+\cos(x)} + \frac{9\cos(x)+5\sin(x)}{\cos(x)+\sin(x)}\,dx = \frac{1}{2}\int_0^{\pi/2} \frac{14\sin(x)+14\cos(x)}{\sin(x)+\cos(x)}\,dx$$

Simplify by factoring and canceling.

$$= \frac{14}{2}\int_0^{\pi/2} \frac{\sin(x)+\cos(x)}{\sin(x)+\cos(x)}\,dx = 7\int_0^{\pi/2} dx = \boxed{\frac{7\pi}{2}}$$

---

19 $\int_0^1 \frac{\ln(x^2+1)}{x}\,dx$

Integrals of this format lead to a function known as the *dilogarithm*. The dilogarithm is defined as:

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\ln(1-t)}{t}\,dt$$

The dilogarithm can be expressed as an infinte sum. First, integrate the geometric series to get a series representation of $-\ln(1-t)$:

$$S(r) = \frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$$

$$\int_0^t S(r)\,dr = -\ln(1-t) = \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{t^n}{n}$$

Divide by t and integrate again to get the desired relation.

$$\text{Li}_2(x) = -\int_0^x \frac{\ln(1-t)}{t}\,dt = \int_0^x \sum_{n=1}^{\infty} \frac{t^{n-1}}{n}\,dt = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Using the sum definition, we can find two special values of the dilogarithm (related to the famous Basel problem).

$$\text{Li}_2(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad\qquad \text{Li}_2(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

To transform the integrand into the dilogarithm, let $u = -x^2$ and $du = -2x\,dx$

$$\int_0^1 \frac{\ln(x^2+1)}{x}\,dx = \int_0^1 \frac{-x\ln(1+x^2)}{-x^2}\,dx = \frac{1}{2}\int_0^{-1} \frac{\ln(1-u)}{u}\,du = -\frac{1}{2}\,\text{Li}_2(-1) = \boxed{\frac{\pi^2}{24}}$$

---

20 Estimate $\displaystyle\int_0^1 \frac{e^{\arctan(x)}}{x+1}\,dx$

Using the over-approximation $\arctan(x) \approx x$, let $f(x) = \dfrac{e^x}{x+1}$. $f(0) = 1$ and $f(1) = \dfrac{e}{2} \approx 1.36$. Because of the influence of e^x, $f(x)$ is always increasing in the interval $0 \le x \le 1$, so the lower bound is 1. We can linearly approximate $f(x)$ and calculate the area of the resulting trapezoid to get an upper bound of 1.18. We over-approximated so the actual value should be closer to the lower bound. Indeed, the actual value to 5 decimal places is $\boxed{1.05447}$.